Double Integrals (Part I)

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1 Double Integrals Over Rectangles

$f(x, y)$ is a two variable function defined on a closed rectangular region $R = [a, b] \times [c, d]$. Suppose $f(x, y) \geq 0$, then the graph of $f(x, y)$ is above the xy-plane. Consider the solid that lies above $R = [a, b] \times [c, d]$ and under the graph of f. We would like to compute its volume.

The strategy is to cut R into many smaller rectangular regions, and approximate the volume of each one by a cuboid, and finally sum them up. When the cutting is finer and finer, the sum of the volume of these small cuboid is closer and closer to the volume of the original solid.

We divide $[a, b]$ into $a = x_0 < x_1 < ... < x_m = b$ and $[c, d]$ into $c = x_0 < x_1 < ... < x_n = d$. The rectangular region R is thus divided into mn regions $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, with area $\Delta A_{ij} = \Delta x_i \Delta y_j$.

For each R_{ij}, we choose a point (x_{ij}^*, y_{ij}^*) as representative point, then the solid above R_{ij} and below the graph of $f(x, y)$ can be approximated by

$$f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$$

and if we sum up all these pieces, the volume of the solid is approximated by

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$$

Now taking the limit as $\max \Delta x_i, \max \Delta y_j \to 0$, if the limit exists, we define it to be the double integral of f over the rectangular region R, and denote it by

$$\iint_R f(x, y)\, dA = \lim_{\max \Delta x_i \to 0, \max \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$$

If we define a more general volume, that is, if $f(x, y)$ is a two variable function (may not be nonnegative), we define the signed volume of the solid bounded between the graph of the function and the xy-plane to be the volume of the part above the xy-plane minus the volume of the part below the xy-plane. In this way we extend the definition of double integral to a general function f, and its value is the signed volume described above.

A function is called integrable if such limit exists.

Theorem 1. *If $f(x, y)$ is bounded and continuous except on a finite number of curves on the region, then f is integrable on this region.*

So many of the two variable functions that we are familiar with are integrable on rectangular regions.

2 Iterated Integrals

$f(x, y)$ is a function on two variables, and continuous on $R = [a, b] \times [c, d]$. Define $\int_c^d f(x, y)\, dy$ to be the integral with y as the variable, regarding x as a constant. The result should be a function of x, denoted by

$$A(x) = \int_c^d f(x, y)\, dy$$

Then we integrate $A(x)$ with respect to x to get

$$\int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy\right) dx$$

This form of integral on the right side is called an **iterated integral**. This is an important construction because of the following theorem:

Theorem 2. *(Fubini's Theorem) If f is a continuous function on a rectangle $R = [a, b] \times [c, d]$, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is also true when $f(x, y)$ is bounded on R and discontinuous only at a finite number of smooth curves.

The proof the the above theorem is very deep in theory, but its intuition is clear:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \lim_{\max \Delta x_i \to 0, \max \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{ij} \\ &= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{m} (\lim_{\max \Delta y_j \to 0} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta y_j) \Delta x_i \\ &= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{m} (\int_c^d f(x_i^*, y)\,dy) \Delta x_i \\ &= \int_a^b (\int_c^d f(x,y)\,dy)\,dx\end{aligned}$$

Example 3. *Evaluate $\iint_R xy\,dA$, where $R = [1,2] \times [3,4]$*

$$\begin{aligned}\iint xy\,dA = \int_1^2 \int_3^4 xy\,dy\,dx &= \int_1^2 (\frac{x}{2} y^2 \Big|_3^4)\,dx \\ &= \int_1^2 \frac{7}{2} x\,dx \\ &= \frac{7}{4} x^2 \Big|_1^2 \\ &= \frac{21}{4}\end{aligned}$$

Proposition 4. *$R = [a,b] \times [c,d]$ is rectangular region.*

1. *If $f(x,y) = g(x)h(y)$, then $\iint_R f(x,y)\,dA = (\int_a^b g(x)\,dx)(\int_c^d h(y)\,dy)$*
2. *$\iint_R f(x,y) + g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$*
3. *$\iint_R cf(x,y)\,dA = c \iint_R f(x,y)\,dA$*
4. *If $f(x,y) \geq g(x,y)$ on R, then $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$*

Example 5. *Evaluate $\iint_R xy\,dA$, where $R = [1,2] \times [3,4]$*

$$\iint xy\,dA = (\int_1^2 x\,dx)(\int_3^4 y\,dy) = \frac{3}{2} \times \frac{7}{2} = \frac{21}{4}$$